

# Wannier-Stark ladders in one-dimensional elastic systems

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The optical analogues of Bloch oscillations and their associated Wannier-Stark ladders have been recently analyzed. In this paper we propose an elastic realization of these ladders, employing for this purpose the torsional vibrations of specially designed one-dimensional elastic systems. We have measured, for the first time, the ladder wave amplitudes, which are not directly accessible either in the quantum mechanical or optical cases. The wave amplitudes are spatially localized and coincide rather well with theoretically predicted amplitudes. The rods we analyze can be used to localize different frequencies in different parts of the elastic systems and viceversa.

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Recently, undulatory systems showing analogues of Bloch oscillations and Wannier-Stark ladders (WSL) attracted increasing attention in several fields of physics [1, 2, 3, 4, 5]. As shown by Bloch, electrons in a periodic potential have extended solutions. The same is true for the behavior of an electron under the action of a static electric field. In contrast, and opposite to intuition, when both the periodic potential and the electric field are present, the solutions are localized; this is only true when band to band Zener tunneling is negligible or the system is short enough. The spectrum then shows equally spaced resonances known as Wannier-Stark ladders, the nearest-neighbor level spacing being proportional to the intensity of the external field [6]. In the time domain, the Wannier-Stark ladders yield the so called Bloch oscillations which consist in a counterintuitive effect where the electrons show an

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oscillatory movement under the action of the static external electric field [7, 8].

The predictions by Bloch and Wannier gave rise to a long controversy that lasted more than 60 years for the Bloch oscillations [9], and more than 20 years for the WSL [10, 11, 12]. The ladders were observed before Bloch oscillations. This was done first in numerical experiments using simple one-dimensional models [13] and later in the laboratory [14]. Bloch oscillations were also observed later on [15]. The most important ingredient to explain WSL is the wavelike behavior of the electrons. Therefore, these ladders could also be observed in classical undulatory systems. Some of these classical systems have been analyzed theoretically [16, 17, 18].

In this paper we study special elastic rods whose torsional waves for free ends present some analogies to the WSL. The first system, depicted in Fig. 1 (a) and which will be referred to as system A, consists of a set of  $N$  circular cylinders of radius  $R$  and varying length  $l_n$ ,  $n = 1, 2, \dots, N$ , separated by very small cylinders of length  $\epsilon \ll l_n$  and radius  $r < R \ll l_n$ . This is the elastic analogue of the optical system with varying widths used in Ref. [1]. System B, shown in Fig. 1 (b), is a beam formed by  $N$  cuboids of constant width  $w$  and length  $l$ . They have different heights  $h_n$ , for  $n = 1, 2, \dots, N$ , with  $w, h_n \ll l$ . These cuboids are separated by small cuboids of dimensions  $h' = w', \epsilon' \ll l$ . This is the elastic analogue of the optical systems with a gradient of the refractive index along the direction of propagation used in Refs. [2, 3, 4], as we shall see below. Systems A and B were constructed by machining a solid aluminum piece.

We first discuss the design of these systems and then, from a qualitative point of view, the normal-mode frequencies and wave amplitudes for torsional vibrations. We later use the transfer matrix method and obtain the normal-mode properties, which will be compared to experimental measurements.

In order to design systems A and B we start with what could be called an independent rod model in which each body oscillates independently from the rest. The normal-mode torsional frequencies  $f_j^{(n)}$  of rod  $n$  with length  $l_n$  and wave velocity  $c_n$  are given by the well-known expression [19]

$$f_j^{(n)} = \frac{c_n}{2l_n} j, \quad (1)$$

where  $j$  is the number of nodes in the wave amplitude. To get a set of equidistant frequencies we consider two options: either the lengths  $l_n$  are varied with a fixed wave velocity or the lengths are kept constant and the wave velocity is changed. For system A we take circular

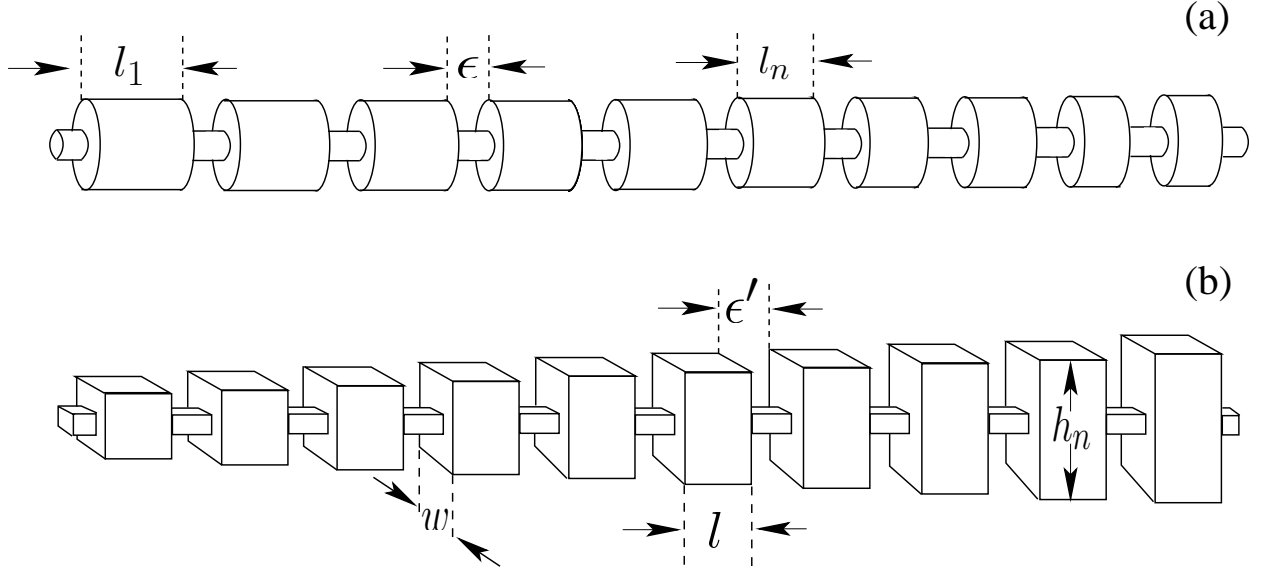


FIG. 1: Rods used to obtain the Wannier-Stark ladders: (a) rod with varying length cells and (b) beam with equal-length cells but different heights. For system A,  $l_n = l/(1 + n\gamma)$ ,  $n = 1, \dots, 14$ , with  $l = 10.8$  cm,  $\gamma = 0.091$ ,  $\epsilon = 2.52$  mm and  $\sqrt{G/\rho} = 3104.7$  m/s. The radii of the small and big cylinders are  $r = 2.415$  mm and  $R = 6.425$  mm, respectively. In system B,  $l = 5.0$  cm,  $w = 1.905$  cm and  $c_n = c(1 + n\gamma)$ ,  $n = 1, \dots, 15$  with  $c = 2027.3$  m/s and  $\gamma = 0.02786$ . The width, height and length of the small cuboids are  $w' = 5.0$  mm,  $h' = 5.0$  mm, and  $\epsilon' = 6.0$  mm, respectively.

rods with  $l_n = l/(1 + n\gamma)$ ,  $n = 1, \dots, N$  and  $c_n = (G/\rho)^{1/2}$  where  $\rho$  is the density,  $G$  the shear modulus and  $l$  a fixed arbitrary length. Notice that, in circular rods, the velocity does not depend on the radius [19]. For system B we take  $l_n = l$  and  $c_n = c(1 + n\gamma)$ ,  $c$  being an arbitrary constant velocity. The parameter  $\gamma$  is dimensionless.

To construct system B we use the Navier expression for the torsional velocity of cuboid  $n$ :

$$c_n = \sqrt{\frac{G\alpha_n}{\rho I_n}} \quad (2)$$

where  $I_n = (h_n w^3 + h_n^3 w)/12$  is the moment of inertia with respect to the axis of the system and  $\alpha_n$  is given by

$$\alpha_n = \frac{256}{\pi^6} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \frac{1}{(2m+1)^2 (2p+1)^2} \times \frac{h_n w}{((2m+1)/h_n)^2 + ((2p+1)/w)^2}. \quad (3)$$

For beams of varying  $h_n$  we have verified Eqs. (2) and (3) experimentally. Solving these equations we have obtained the values of  $h_n$  such that  $c_n = c(1 + n\gamma)$ .

Then

$$f_j^{(n)} = \begin{cases} \sqrt{G/\rho}(1 + n\gamma)j/2l & \text{for system A} \\ c(1 + n\gamma)j/2l & \text{for system B,} \end{cases} \quad (4a)$$

$$(4b)$$

and the differences  $\Delta f_j^{(n)} = f_j^{(n+1)} - f_j^{(n)}$  are equal to

$$\Delta_j \equiv \Delta f_j^{(n)} = \begin{cases} \sqrt{G/\rho}\gamma j/2l & \text{for system A} \\ c\gamma j/2l & \text{for system B,} \end{cases} \quad (5a)$$

$$(5b)$$

which are independent of index  $n$ .

We shall now discuss system A. When the arbitrary parameter  $\gamma$  is set equal to zero, a locally periodic rod is formed. This locally periodic rod shows a band spectrum [20]. When  $\gamma \neq 0$  a completely different spectrum occurs. The new spectrum then resembles the Wannier–Stark ladder.

Before presenting the calculation of the normal modes for this system, and then showing numerical and experimental results, let us make a qualitative analysis to see what type of spectrum could be expected from the independent rod model. At the lowest frequencies, the wavelength  $\lambda$  is of the same order of magnitude as  $L \approx \sum_{n=1}^N l_n$ , and the whole rod is excited. But when  $\lambda$  decreases and becomes of the order of  $l_1 = l/(1 + \gamma)$ , the longest rod, say rod 1, is excited in a state equivalent to its lowest normal mode. The rest of the  $N$  rods are out of resonance, so the amplitude decreases as we move farther away from rod 1. Therefore, the state is localized around the latter. In some sense this was to be expected since we are disturbing a periodic structure to obtain a disordered one-dimensional system, which always shows localized wave amplitudes. Increasing the exciting frequency by  $\Delta_1$  the rod with length  $l_2 = l/(1 + 2\gamma)$ , that is, rod 2, will now be excited and the rest will be out of resonance. The amplitudes of the vibrations therefore decrease as their distance from rod 2 increases. The wave amplitude is again localized but now around rod 2; it has a similar shape as the wave amplitude that rod 1 had before, but it has been slightly deformed, squeezed, and translated from rod 1 to rod 2. The same argument applies when rod  $n$  of length  $l_n = l/(1 + n\gamma)$  is excited.

What we have done is to produce a finite WSL, i.e.,  $N$  localized states with constant difference in frequency given by Eq. (5a). However, more ladders exist since normal modes

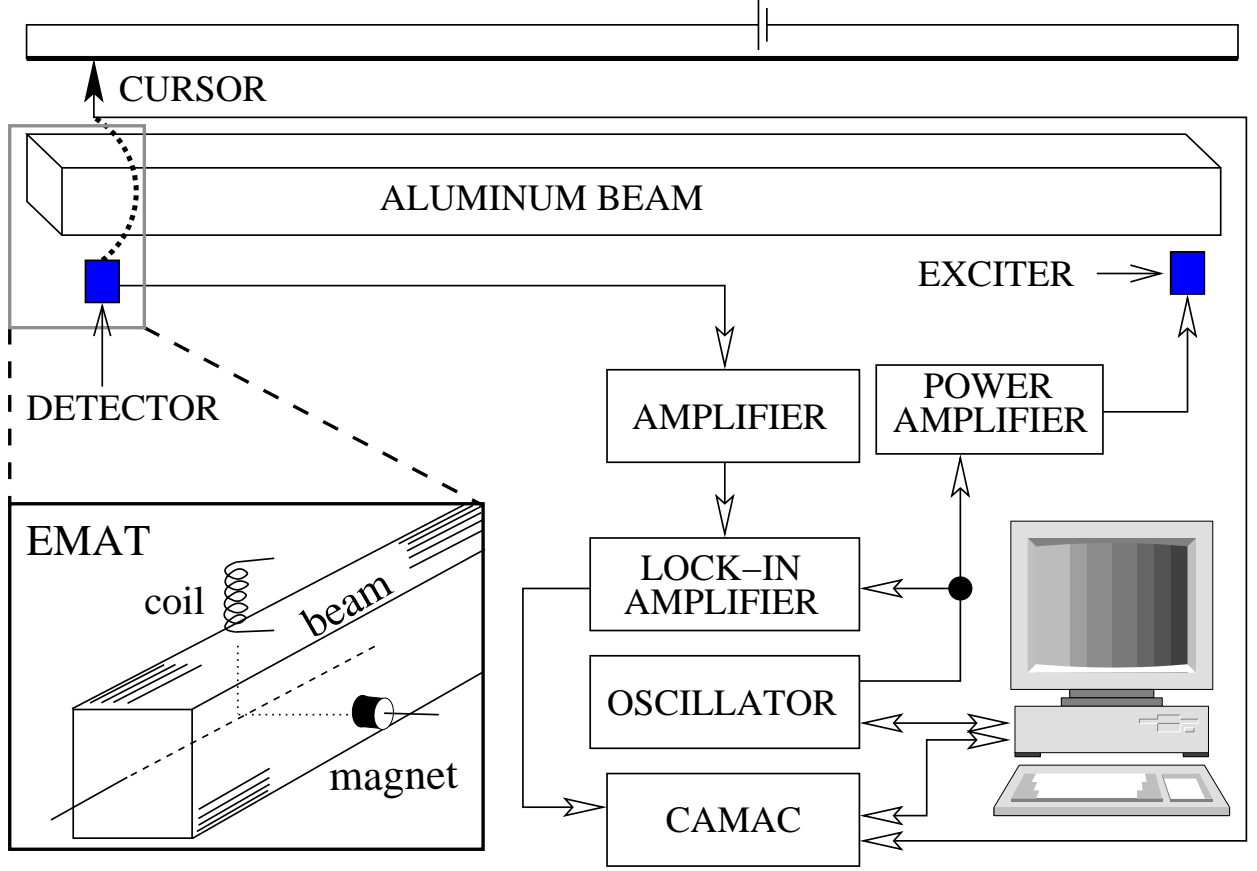


FIG. 2: Block diagram of the experimental setup. For system A both detector and exciter are EMATs, while for system B the exciter was a piezoelectric transducer.

with two or more nodes can also be excited in each rod. For instance, taking  $j = 2$  in Eq. (4a) a second ladder is obtained. This ladder is different from the first one because the frequency difference is now twice the one of the lower WSL, as can be seen from Eq. (5a). The states are again localized and all have a similar shape although squeezed. A third ladder exists with  $\Delta_3 = 3\Delta_1$  and so on for other values of  $j$ . A similar argument for system B shows also the existence of several WSL. The difference between the quantum-mechanical and elastic ladders is that in the latter the spacing between resonances is not the same for different ladders.

We have calculated the eigenmode properties of the rods of Fig. 1 (a) and (b) with free-end boundary conditions using the transfer matrix method for torsional waves discussed in Ref. [20]. The normal mode frequencies and amplitudes were measured using the experimental set up described in Fig. 2. We use an electromagnetic acoustic transducer (EMAT)

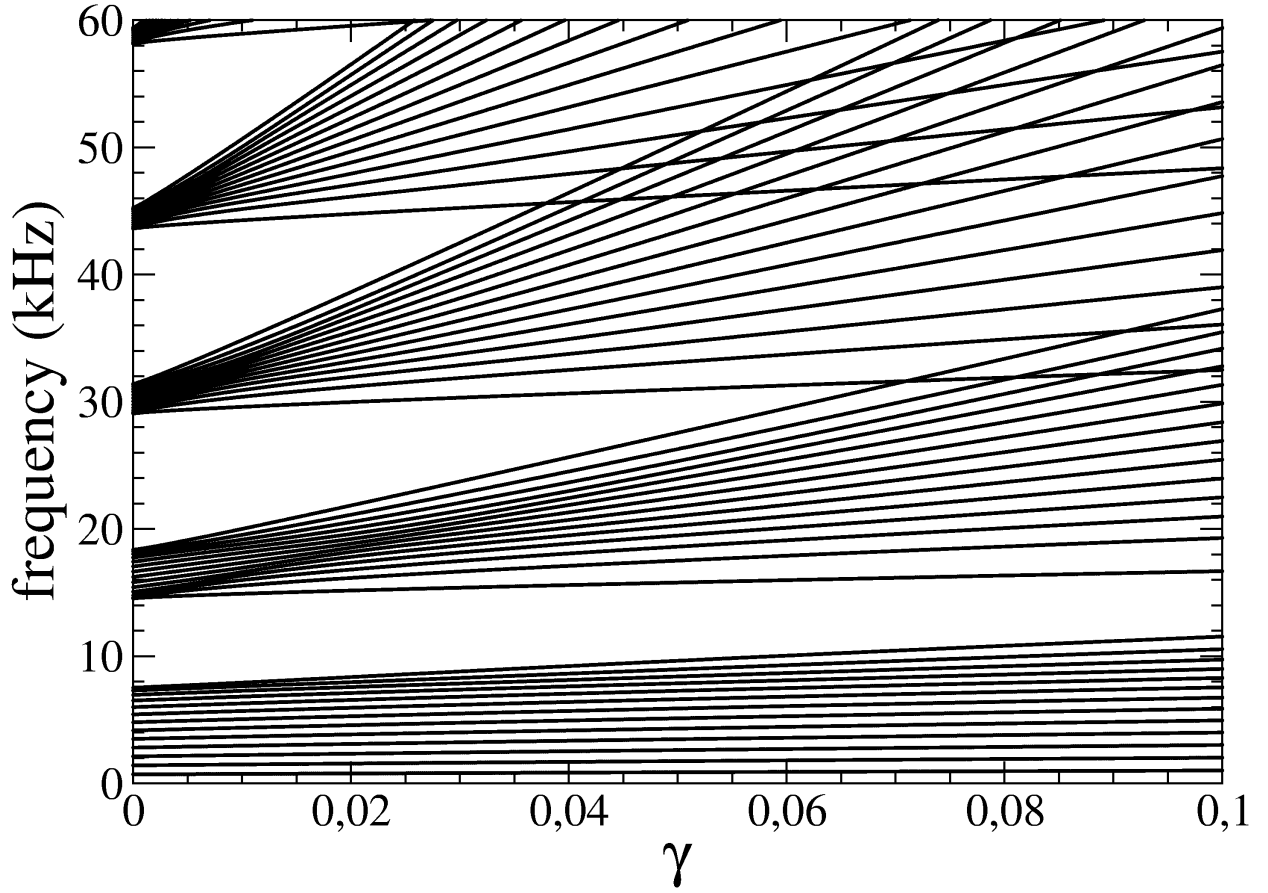


FIG. 3: Normal-mode frequencies of System A as a function of the dimensionless parameter  $\gamma$ .

which is very versatile and operates at low frequencies. This EMAT, which we have recently developed [21], can selectively excite or detect compressional, torsional or flexural vibrations. In the inset of Fig. 2 the EMAT has been installed to detect torsional vibrations; see Ref. [20].

We show in Fig. 3 the spectrum of system A as a function of the dimensionless parameter  $\gamma$ . As mentioned above, for  $\gamma = 0$  a band spectrum appears, and as  $\gamma$  grows the levels of each band separate to form the WSL. The normal-mode frequencies of the rod shown in Fig. 1 (a) are given in Fig. 4 for  $\gamma = 0.091$ . We first note that the theoretical results coincide very well with the experimental ones. Furthermore, the qualitative treatment provides a rather good first approximation. One can see from this figure that the states form a set of Wannier-Stark ladders, as discussed before. The first band composed by the extended modes is not displayed in this graph. Notice that the frequencies in the extremes of each ladder do not have the same difference in frequency as those at the middle of the ladder.

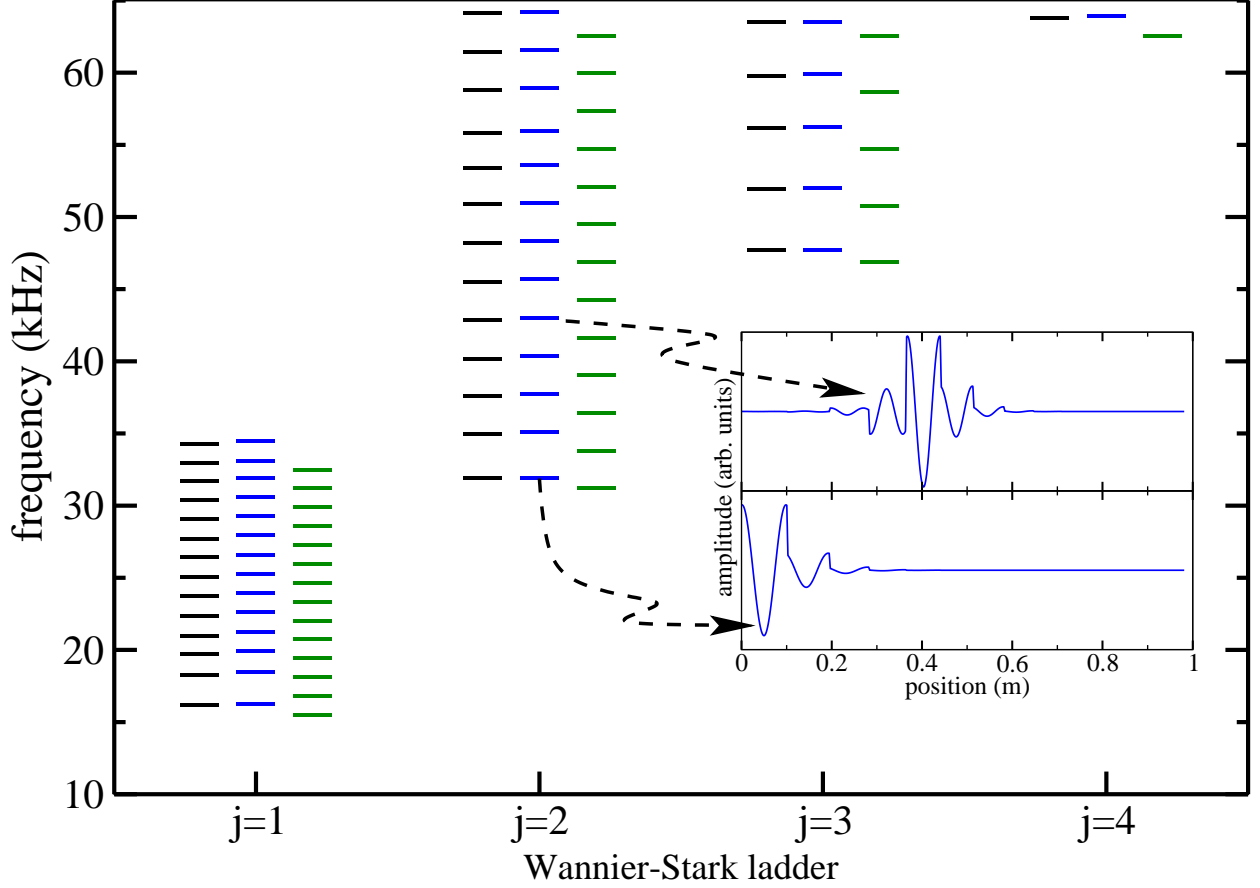


FIG. 4: (color online) Normal-mode frequencies of System A yielding the elastic Wannier-Stark effect. For each value of  $j$  the left hand side column corresponds to the experimental values, the middle column to the numerical results obtained using the transfer matrix method and the right hand side shows the approximate results following from the independent rod model. In the calculation we used an effective value for  $r/R$  [20]. The uncertainty in the experimental values is less than 0.01%. In the insets the theoretical wave amplitudes are given for a state at the extreme of the ladder and another one in the center of it.

This is due to a border effect in the wave amplitudes localized near the free ends. As shown in the insets of Fig. 4 the border amplitude lacks a portion of the wave amplitude that the states at the center of the ladder have.

In Fig. 5 we show the comparison of theoretical and experimental wave amplitudes. These are localized around rod  $n$ . For example, in Fig. 5 (a) the sixth rod resonates and in Fig. 5 (b) another state corresponding to the same ladder is localized around the tenth rod. Both have the same form although squeezed. Figures 5 (c) and 5 (d) show two states of the second

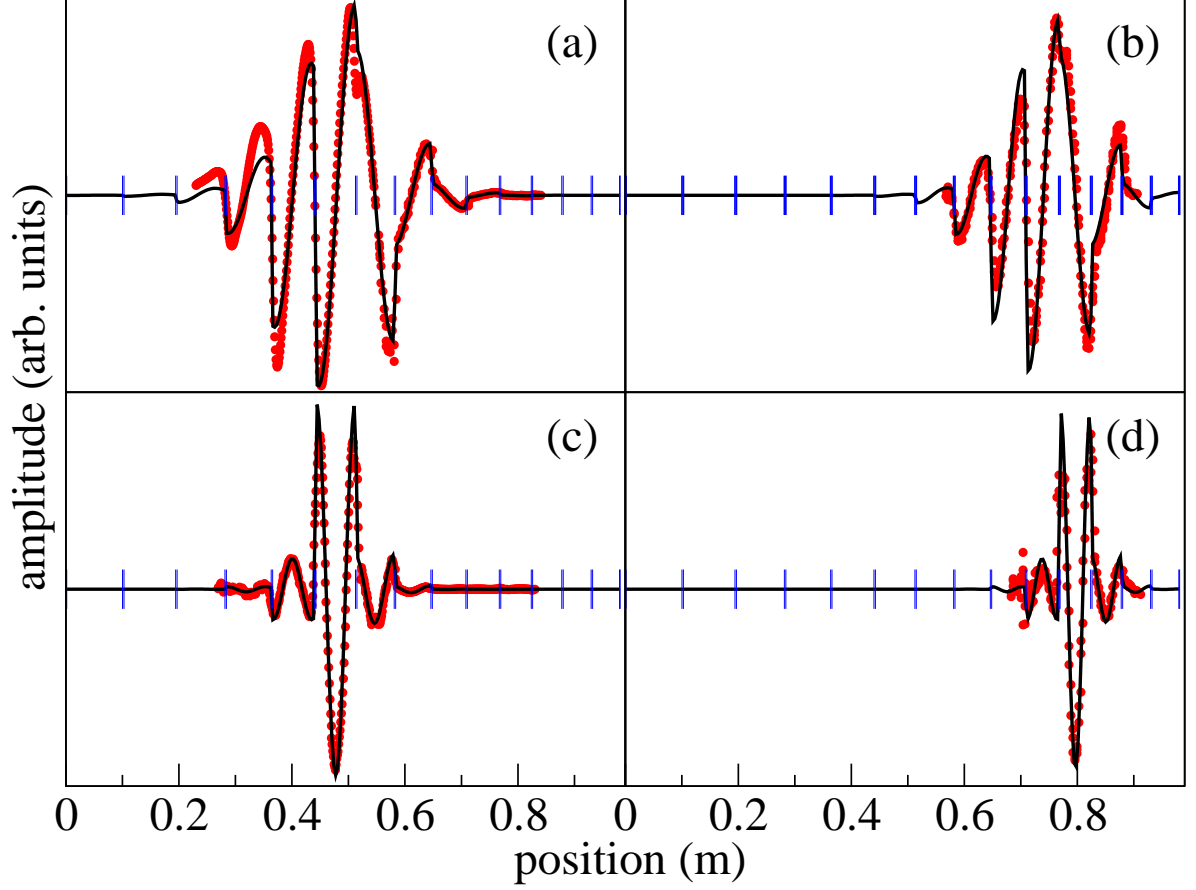


FIG. 5: (color online) Wave amplitudes of system A for 19 nodes (a), 23 nodes (b), 33 nodes (c), and 41 nodes (d). The continuous line corresponds to the transfer-matrix results and the dots to the measurements. The vertical lines along the rod axis indicate the position of the notches.

ladder, with  $n = 6$  and  $n = 11$ , respectively. Localization is again observed and, as was to be expected, the amplitudes now show two nodes in the resonating rods. Note the excellent agreement between theory and experiment after adjusting the height of the theoretical wave amplitude at only one point.

As mentioned above, system B shows similar properties. We present in Fig. 6, as an example, two wave amplitudes for the first WSL of system B. In contrast with system A, these wave amplitudes have the same shape but translated and are not squeezed. Notice that the one-dimensional transfer matrix method fit the experimental wave amplitudes in spite of the fact that  $w$  and  $h_n$  are not much smaller than  $l$ .



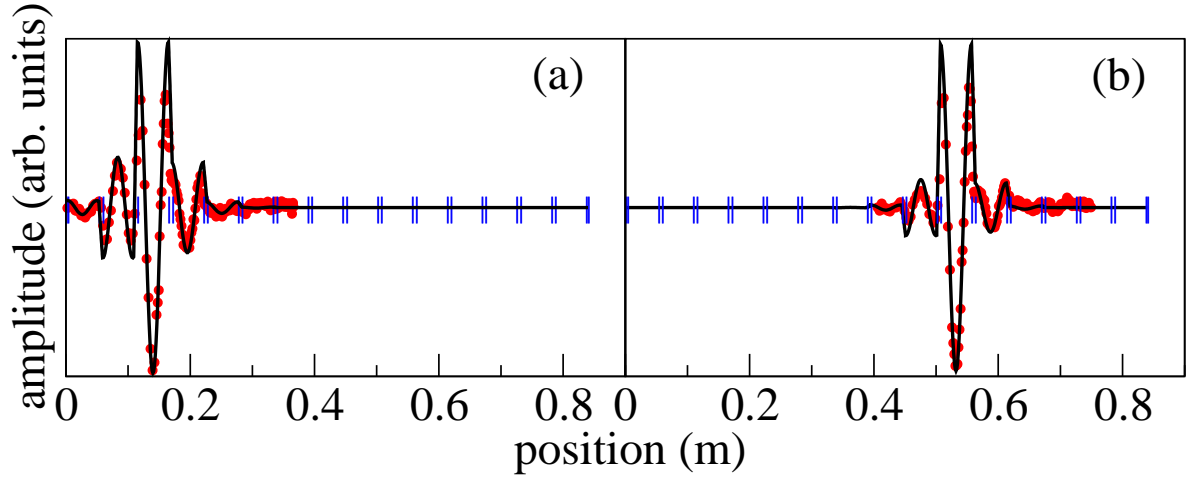


FIG. 6: (color online) Two wave amplitudes of system B, (a) one localized on the third rod with frequency  $f = 44.256$  kHz and (b) other localized on the tenth rod with frequency  $f = 51.258$  kHz. The double vertical lines along the beam axis indicate the position of the notches.

To conclude, we have constructed an elastic analogue of Wannier–Stark ladders. In contrast with the optical analogue of Refs. [1, 2, 3, 4] we have observed the WSL directly. Furthermore, we measured for the first time the wave amplitudes, including phases, which show localization. We also observed higher Wannier–Stark ladders. The elastic Wannier–Stark ladders have potential applications in the design of systems with localized vibrations.

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